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Distributed order derivatives and relaxation patterns

Anatoly N Kochubei

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska 3, Kiev 01601, Ukraine

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Abstract
We consider equations of the form

$$(\mathbb{D}_{(\rho)}u)(t) = -\lambda u(t), \qquad t > 0,$$

where $\lambda > 0$, $\mathbb{D}_{(\rho)}$ is a distributed order derivative, that is

$$\mathbb{D}_{(\rho)}\varphi(t) = \int_0^1 (\mathbb{D}^{(\alpha)}\varphi)(t)\,\mathrm{d}\rho(\alpha),$$

where $\mathbb{D}^{(\alpha)}$ is the Caputo–Dzhrbashyan fractional derivative of order α, ρ is a positive measure.

The above equation is used for modeling anomalous, non-exponential relaxation processes. In this work, we study the asymptotic behavior of solutions of the above equation, depending on the properties of the measure ρ.

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1. Introduction

Anomalous, non-exponential, relaxation processes occur in various branches of physics; see, e.g., [3, 18, 21] and references therein. Just as the exponential function $\mathrm{e}^{-\lambda t}$ ($\lambda > 0$) appearing in the description of classical relaxation is a solution of the simplest differential equation, $u' = -\lambda u$ (with the initial condition $u(0) = 1$), new kinds of derivatives are used to obtain models of slow relaxation.

It is now generally accepted that the power law of relaxation corresponds to the Cauchy problem:

$$(\mathbb{D}^{(\alpha)}u)(t) = -\lambda u(t), \qquad t > 0, \quad u(0) = 1, \tag{1}$$

where

$$(\mathbb{D}^{(\alpha)}u)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\tau)^{-\alpha}u(\tau)\,\mathrm{d}\tau - t^{-\alpha}u(0)\right] \tag{2}$$

is the Caputo–Dzhrbashyan fractional derivative of order $\alpha \in (0, 1)$; we refer to [7, 13] for various notions and results regarding fractional differential equations. The solution of problem (1) has the form $u(t) = E_\alpha(-\lambda t^\alpha)$, where E_α is the Mittag–Leffler function, and $u(t) \sim Ct^{-\alpha}$

(here and below we denote various positive constants by the same letter C), as $t \to \infty$. This kind of evolution describes the temporal behavior related to α-fractional diffusion typical for fractal media.

A still slower, logarithmic relaxation [11, 16] is described by the Cauchy problem:

$$(\mathbb{D}^{(\mu)}u)(t) = -\lambda u(t), \qquad t > 0, \quad u(0) = 1, \tag{3}$$

where

$$(\mathbb{D}^{(\mu)}u)(t) = \int_0^1 (\mathbb{D}^{(\alpha)}u)(t)\mu(\alpha)\,\mathrm{d}\alpha, \tag{4}$$

where μ is a non-negative continuous function on $[0, 1]$. A rigorous mathematical treatment of problem (3) for general classes of the weights μ was given in [14, 15]. Some nonlinear equations with distributed order derivatives were studied in [1]. As above, this kind of relaxation models is connected with models of ultraslow diffusion [4, 10, 12, 14, 17, 22]; these papers contain further references on related subjects.

Let $u_\lambda(t)$ be the solution of problem (3) (the notation is changed slightly, compared to [14]). If $\mu(0) \neq 0$, then

$$u_\lambda(t) \sim C(\log t)^{-1}, \qquad t \to \infty. \tag{5}$$

If $\mu(\alpha) \sim a\alpha^\nu$, $\alpha \to 0$ ($a > 0$, $\nu > 0$), then

$$u_\lambda(t) \sim C(\log t)^{-1-\nu}, \qquad t \to \infty. \tag{6}$$

It was assumed everywhere in [14] that $\mu \in C^3[0, 1]$ and $\mu(1) \neq 0$; in fact, in the investigation of the asymptotic behavior of u_λ we use only that $\mu \in L_1(0, 1)$. Therefore, the arguments in [14] cover the case, where

$$\mu(\alpha) \sim a\alpha^{-\nu}, \qquad \alpha \to 0,$$

with $a > 0$, $0 < \nu < 1$, and yield the asymptotics

$$u_\lambda(t) \sim C(\log t)^{-1+\nu}, \qquad t \to \infty. \tag{7}$$

In all the above cases, the exponential function $\mathrm{e}^{-\lambda t}$, the Mittag–Leffler powerlike evolution $E_\alpha(-\lambda t^\alpha)$ and all the logarithmic evolutions (5)–(7), the resulting functions are completely monotone, that is $(-1)^j u_\lambda^{(j)}(t) \geqslant 0$, $j = 0, 1, 2, \ldots$, for all t.

In this paper, we look for other possible relaxation patterns corresponding to the weight functions μ tending to 0 (at the origin) faster than the power function (section 2) or to the definition of the distributed order derivative not by formula (4) but by the expression

$$(\mathbb{D}_{(\rho)}u)(t) = \int_0^1 (\mathbb{D}^{(\alpha)}u)(t)\,\mathrm{d}\rho(\alpha), \tag{8}$$

where ρ is a jump measure (section 3). In these cases, the solutions remain completely monotone while their asymptotic behavior can be quite diverse, from the iterated logarithmic decay to a function decaying faster than any power of logarithm but slower than any power function.

2. The main constructions

Suppose that ρ is a positive finite measure on $[0, 1]$ not concentrated at 0, and consider the distributed order derivative (8). Substituting (2) into (8), we find that

$$(\mathbb{D}_{(\rho)}u)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(t-\tau)u(\tau)\,\mathrm{d}\tau - k(t)u(0), \tag{9}$$

where

$$k(s) = \int_0^1 \frac{s^{-\alpha}}{\Gamma(1-\alpha)}\, d\rho(\alpha), \qquad s > 0. \tag{10}$$

The right-hand side of (9) makes sense for a continuous function u, for which the derivative $\frac{d}{dt}\int_0^t k(t-\tau)u(\tau)\, d\tau$ exists.

It is clear from (10) that $k \in L_1^{\mathrm{loc}}(0,\infty)$, and k is decreasing. Therefore, the function k possesses the Laplace transform:

$$\mathcal{K}(p) = \int_0^\infty k(s)\, e^{-ps}\, ds = \int_0^1 p^{\alpha-1}\, d\rho(\alpha), \qquad \operatorname{Re} p > 0.$$

The holomorphic function, $\mathcal{K}(p)$, can be extended analytically onto the whole complex plane cut along the half-axis $\mathbb{R}_- = \{\operatorname{Im} p = 0, \operatorname{Re} p \leqslant 0\}$. Obviously, $\mathcal{K}(p) \to 0$, as $|p| \to \infty$ (a precise asymptotics is found for some cases in [14] and section 3 below).

Note that if we consider the moments of the measure ρ,

$$M_n = \int_0^1 x^n\, d\rho(x),$$

and introduce their generating function,

$$M(z) = \sum_{n=0}^\infty M_n \frac{z^n}{n!} = \int_0^1 e^{xz}\, d\rho(x),$$

then $\mathcal{K}(p) = p^{-1}M(\log p)$. The moment generating functions were studied by a number of authors (for example, [5, 20]).

Considering the relaxation equation,

$$(\mathbb{D}_{(\rho)}u)(t) = -\lambda u(t), \qquad t > 0, \tag{11}$$

with $\lambda > 0$, we apply formally the Laplace transform (which is justified *post factum*, using the smoothness properties and the asymptotic behavior of the solution).

For the Laplace transform $\widetilde{u_\lambda}(p)$ of the solution $u_\lambda(t)$ of equation (11) satisfying the initial condition $u_\lambda(0) = 1$, we get the expression

$$\widetilde{u_\lambda}(p) = \frac{\mathcal{K}(p)}{p\mathcal{K}(p) + \lambda}. \tag{12}$$

Since $p\mathcal{K}(p) \to \infty$, as $p \to \infty$, we have $\widetilde{u_\lambda}(p) \sim p^{-1}$, $p = \sigma + i\tau, \sigma, \tau \in \mathbb{R}, |\tau| \to \infty$. Therefore [6], $\widetilde{u_\lambda}$ is the Laplace transform of some function $u_\lambda(t)$, and for almost all t,

$$u_\lambda(t) = \frac{d}{dt}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}}{p}\frac{\mathcal{K}(p)}{p\mathcal{K}(p)+\lambda}\, dp. \tag{13}$$

Note that for $p \in \mathbb{C}\backslash\mathbb{R}_-$ we have

$$\operatorname{Im} p\mathcal{K}(p) = \int_0^1 |p|^\alpha \sin(\alpha \arg p)\, d\rho(\alpha),$$

so that $\operatorname{Im} p\mathcal{K}(p) = 0$ only for $\arg p = 0$. This means that $p\mathcal{K}(p) + \lambda \neq 0$, and representation (13) is valid for an arbitrary $\gamma > 0$.

As in [14], in the present more general situation we deform the contour of integration, and then differentiate under the integral, so that

$$u_\lambda(t) = \frac{1}{2\pi i}\int_{S_{\gamma,\omega}} e^{pt}\frac{\mathcal{K}(p)}{p\mathcal{K}(p)+\lambda}\, dp, \tag{14}$$

where the contour $S_{\gamma,\omega}$ consists of the arc,

$$T_{\gamma,\omega} = \{p \in \mathbb{C} : |p| = \gamma, |\arg p| \leqslant \omega\pi\}, \qquad \tfrac{1}{2} < \omega < 1,$$

and two rays,

$$\Gamma^{\pm}_{\gamma,\omega} = \{p \in \mathbb{C} : |\arg p| = \pm\omega\pi, |p| \geqslant \gamma\}.$$

Just as in the case of a measure ρ with a smooth density considered in [14], it is easy to show that the function u_λ belongs to $C^\infty(0,\infty)$ and is continuous at the origin; from its construction and formula (9), it follows that the initial condition, $u_\lambda(0) = 1$, is indeed satisfied.

Let us consider first the case of a measure ρ with a continuous density, $\mathrm{d}\rho(\alpha) = \mu(\alpha)\,\mathrm{d}\alpha$. This case was investigated in [14], and was proved that u_λ *is completely monotone* (various additional assumptions made in [14] and needed for other problems studied in that paper were not actually used here).

In this paper, we consider the case of a different behavior of the density μ near the origin, implying a different asymptotics of $u_\lambda(t)$, as $t \to \infty$.

Theorem 1. *If* $\mu \in C[0,1]$ *and*

$$\mu(\alpha) \sim a\alpha^\gamma\,\mathrm{e}^{-\frac{\beta}{\alpha}}, \qquad \text{as} \quad \alpha \to 0,$$

where $a > 0, \gamma > -1, \beta > 0$, *then*

$$u_\lambda(t) \sim C(\log t)^{-\frac{\gamma}{2}-\frac{3}{4}}\,\mathrm{e}^{-2\sqrt{\beta}(\log t)^{\frac{1}{2}}}, \qquad t \to \infty. \tag{15}$$

Proof. Let us write $\mathcal{K}(p)$ as

$$\mathcal{K}(p) = p^{-1}\int_0^\infty \mathrm{e}^{-\alpha z}\mu_1(\alpha)\,\mathrm{d}\alpha, \qquad z = \log\frac{1}{p},$$

where μ_1 is the extension of μ by zero onto $\mathbb{R}_+$, and use an asymptotic result for Laplace integrals from [19] (theorem 13.1, case 9). We get

$$\mathcal{K}(p) \sim 2ap^{-1}\left(\frac{\beta}{z}\right)^{\frac{\gamma+1}{2}} K_{\gamma+1}(2\sqrt{\beta z}), \qquad p \to +0,$$

where, as before, $z = \log\frac{1}{p}$ $(\to \infty)$, K_m is the McDonald function. It is well known that $K_m(t) \sim \left(\frac{\pi}{2}\right)^{1/2} t^{-1/2}\,\mathrm{e}^{-t}$, $t \to \infty$, so that

$$\mathcal{K}(p) \sim Cp^{-1}L\left(\frac{1}{p}\right), \qquad p \to +0, \tag{16}$$

where

$$L(s) = (\log s)^{-\frac{\gamma}{2}-\frac{3}{4}}\,\mathrm{e}^{-2\sqrt{\beta}(\log s)^{\frac{1}{2}}}.$$

It follows from (16) (or directly from the definition of $\mathcal{K}(p)$) that $p\mathcal{K}(p) \to 0$, as $p \to +0$. Therefore, by (12),

$$\widetilde{u_\lambda}(p) \sim \lambda^{-1}\mathcal{K}(p), \qquad \text{as} \quad p \to +0.$$

Since we have already known that the function u_λ is monotone, we may apply the Karamata–Feller Tauberian theorem (see chapter XIII in [9]) which implies the desired asymptotics of $u_\lambda(t)$, $t \to \infty$. □

In the case under consideration, the function $u_\lambda(t)$ decreases at infinity slower than any negative power of t, but faster than any negative power of $\log t$. It is also seen from (15) that the decrease is accelerated if $\beta > 0$ becomes bigger, so that the less the weight function μ is near 0, the faster is the relaxation for large times.

3. The step Stieltjes weight

Let us consider the case where the integral in (8) is a Stieltjes integral corresponding to a right continuous non-decreasing step function $\rho(\alpha)$. In order to investigate a sufficiently general situation, we assume that the function ρ has two sequences of jump points, β_n and $\nu_n, n = 0, 1, 2, \ldots$, where $\beta_n \to 0, \nu_n \to 1, \beta_0 = \nu_0 \in (0, 1)$. We may assume that the sequence $\{\beta_n\}$ is strictly decreasing while $\{\nu_n\}$ is strictly increasing.

Denote $\Delta\rho(t) = \rho(t) - \rho(t-0), \xi_n = \Delta\rho(\beta_n)\ (n \geqslant 0), \eta_n = \Delta\rho(\nu_n)\ (n \geqslant 1)$; we have $\xi_n, \eta_n > 0$ for all n. It will be convenient to assume that $\beta_0 < \mathrm{e}^{-1}$ and to write $\eta_0 = 0$. Since ρ is a finite measure, we have also

$$\sum_{n=0}^{\infty} \xi_n < \infty, \qquad \sum_{n=0}^{\infty} \eta_n < \infty. \tag{17}$$

By (10),

$$k(s) = \sum_{n=0}^{\infty} \frac{\xi_n}{\Gamma(1-\beta_n)} s^{-\beta_n} + \sum_{n=1}^{\infty} \frac{\eta_n}{\Gamma(1-\nu_n)} s^{-\nu_n}, \qquad s > 0,$$

so that

$$\mathcal{K}(p) = \sum_{n=0}^{\infty} \xi_n p^{\beta_n - 1} + \sum_{n=1}^{\infty} \eta_n p^{\nu_n - 1}. \tag{18}$$

As before, we denote by $u_\lambda(t)$ the solution of the relaxation equation (11) with the initial condition $u_\lambda(0) = 1$. The symbol $f \asymp g$ will, as usual, mean that $f = O(g)$ and $g = O(f)$.

Theorem 2. (i) *The function u_λ is completely monotone.*

(ii) $$u_\lambda(x) \asymp \sum_{n=0}^{\infty} \left[\frac{\xi_n}{\Gamma(2-\beta_n)} x^{-\beta_n} + \frac{\eta_n}{\Gamma(2-\nu_n)} x^{-\nu_n} \right], \qquad x \to \infty. \tag{19}$$

(iii) *If* $\sum_{n=0}^{\infty} \xi_n \left(\log\log \frac{1}{\beta_n}\right)^b < \infty\ (b > 0)$, *then*

$$u_\lambda(x) = O\left(\frac{1}{(\log\log x)^b}\right), \qquad x \to \infty. \tag{20}$$

(iv) *If* $\sum_{n=0}^{\infty} \xi_n \beta_n^{-b} < \infty\ (b > 0)$, *then*

$$u_\lambda(x) = O\left(\frac{1}{(\log x)^b}\right), \qquad x \to \infty. \tag{21}$$

Proof. Using representation (14) and following [14], we can write

$$\begin{aligned} u_\lambda(t) = {} & \frac{1}{2\pi \mathrm{i}} \int_{T_{\gamma,\omega}} \mathrm{e}^{pt} \frac{\mathcal{K}(p)}{p\mathcal{K}(p) + \lambda} \, \mathrm{d}p + \frac{1}{\pi} \operatorname{Im} \int_{\gamma}^{\infty} r^{-1} \mathrm{e}^{tr\,\mathrm{e}^{\mathrm{i}\omega\pi}} \, \mathrm{d}r \\ & - \frac{\lambda}{\pi} \operatorname{Im} \int_{\gamma}^{\infty} \frac{\mathrm{e}^{tr\,\mathrm{e}^{\mathrm{i}\omega\pi}}}{r\,(r\,\mathrm{e}^{\mathrm{i}\omega\pi}\mathcal{K}(r\,\mathrm{e}^{\mathrm{i}\omega\pi}) + \lambda)} \mathrm{d}r \stackrel{\mathrm{def}}{=} J_1 + J_2 - J_3. \end{aligned}$$

Let us substantiate passing to the limit as $\gamma \to 0$.

As $p \to 0$, $p\mathcal{K}(p) \to 0$, so that

$$\left| \frac{\mathcal{K}(p)}{p\mathcal{K}(p) + \lambda} \right| \leqslant C \sum_{n=0}^{\infty} (\xi_n p^{\beta_n - 1} + \eta_n p^{\nu_n - 1}),$$

whence

$$|J_1| \leqslant C\,\mathrm{e}^{\gamma t}\sum_{n=0}^{\infty}(\xi_n\gamma^{\beta_n}+\eta_n\gamma^{\nu_n}) \to 0,$$

as $\gamma \to 0$. It was shown in [14] that

$$J_2 \longrightarrow -\frac{1}{\pi}\int_0^\infty s^{-1}\,\mathrm{e}^{-s}\sin(s\tan\omega\pi)\,\mathrm{d}s,$$

as $\gamma \to 0$.

The integral J_3 is the sum of

$$I_1 = \frac{\lambda}{\pi}\int_\gamma^\infty \mathrm{Im}\left(\frac{\mathrm{e}^{tr\,\mathrm{e}^{\mathrm{i}\omega\pi}}}{r}\right)\mathrm{Re}\left(\frac{1}{r\,\mathrm{e}^{\mathrm{i}\omega\pi}\mathcal{K}(r\,\mathrm{e}^{\mathrm{i}\omega\pi})+\lambda}\right)\mathrm{d}r$$

and

$$I_2 = \frac{\lambda}{\pi}\int_\gamma^\infty \mathrm{Re}\left(\frac{\mathrm{e}^{tr\,\mathrm{e}^{\mathrm{i}\omega\pi}}}{r}\right)\mathrm{Im}\left(\frac{1}{r\,\mathrm{e}^{\mathrm{i}\omega\pi}\mathcal{K}(r\,\mathrm{e}^{\mathrm{i}\omega\pi})+\lambda}\right)\mathrm{d}r.$$

We have

$$\mathrm{Im}\left(\frac{\mathrm{e}^{tr\,\mathrm{e}^{\mathrm{i}\omega\pi}}}{r}\right) = r^{-1}\,\mathrm{e}^{tr\cos\omega\pi}\sin(tr\sin\omega\pi),$$

and this expression has a finite limit, as $r \to 0$. Since also $p\mathcal{K}(p) \to 0$, as $p \to 0$, we see that we may pass to the limit in I_1, as $\gamma \to 0$.

Let

$$\Phi(r,\omega) = \mathrm{Im}\,\frac{1}{r\,\mathrm{e}^{\mathrm{i}\omega\pi}\mathcal{K}(r\,\mathrm{e}^{\mathrm{i}\omega\pi})+\lambda}.$$

Substituting (18) and denoting

$$G(r,\omega) = \left\{\sum_{n=0}^{\infty}[\xi_n r^{\beta_n}\cos(\omega\pi\beta_n)+\eta_n r^{\nu_n}\cos(\omega\pi\nu_n)]+\lambda\right\}^2 + \left\{\sum_{n=0}^{\infty}[\xi_n r^{\beta_n}\sin(\omega\pi\beta_n)+\eta_n r^{\nu_n}\sin(\omega\pi\nu_n)]\right\}^2$$

we find that

$$\Phi(r,\omega) = -\frac{\sum_{n=0}^{\infty}[\xi_n r^{\beta_n}\sin(\omega\pi\beta_n)+\eta_n r^{\nu_n}\sin(\omega\pi\nu_n)]}{G(r,\omega)}.$$

The denominator tends to λ^2, as $r \to 0$. Noting that

$$\mathrm{Re}\left(\frac{\mathrm{e}^{tr\,\mathrm{e}^{\mathrm{i}\omega\pi}}}{r}\right) = r^{-1}\,\mathrm{e}^{tr\cos\omega\pi}\cos(tr\sin\omega\pi)$$

and using (17) we find that the integrand in I_2 belongs to $L_1(0,\infty)$.

Passing to the limit $\gamma \to 0$ we obtain the representation

$$u_\lambda(t) = -\frac{1}{\pi}\int_0^\infty s^{-1}\,\mathrm{e}^{-s}\sin(s\tan\omega\pi)\,\mathrm{d}s - \frac{\lambda}{\pi}\int_0^\infty r^{-1}\,\mathrm{e}^{tr\cos\omega\pi}\sin(tr\sin\omega\pi)\Psi(r,\omega)\,\mathrm{d}r - \frac{\lambda}{\pi}\int_0^\infty r^{-1}\,\mathrm{e}^{tr\cos\omega\pi}\cos(tr\sin\omega\pi)\Phi(r,\omega)\,\mathrm{d}r$$

where

$$\Psi(r,\omega) = \frac{\sum_{n=0}^{\infty}[\xi_n r^{\beta_n}\cos(\omega\pi\beta_n)+\eta_n r^{\nu_n}\cos(\omega\pi\nu_n)]+\lambda}{G(r,\omega)}.$$

Next, let us pass to the limit, as $\omega \to 1$. It follows from the Lebesgue theorem that the first two integrals tend to zero, and we get

$$u_\lambda(t) = \frac{\lambda}{\pi}\int_0^\infty r^{-1}\,\mathrm{e}^{-tr}\,\frac{\sum_{n=0}^\infty [\xi_n r^{\beta_n}\sin(\pi\beta_n) + \eta_n r^{\nu_n}\sin(\pi\nu_n)]}{G(r,1)}\,\mathrm{d}r,$$

that is, up to a positive factor, u_λ is the Laplace transform of a locally integrable non-negative function bounded at infinity. Therefore, u_λ is completely monotone.

(ii) It will be convenient to turn to the Laplace–Stieltjes transform instead of the Laplace transform. Set

$$\varkappa(x) = \int_0^x k(s)\,\mathrm{d}s, \qquad v_\lambda(x) = \int_0^x u_\lambda(s)\,\mathrm{d}s,$$

so that the Laplace–Stieltjes transforms are as follows:

$$\widehat{\varkappa}(p) = \int_0^\infty \mathrm{e}^{-px}\,\mathrm{d}\varkappa(x) = \mathcal{K}(p), \qquad \widehat{v_\lambda}(p) = \widetilde{u_\lambda}(p).$$

By (12), $\widehat{v_\lambda}(p) = l(p)\widehat{\varkappa}(p)$ where $l(p) = \frac{1}{p\mathcal{K}(p)+\lambda}$ is a slowly varying function near the origin.

We have

$$\varkappa(x) = \sum_{n=0}^\infty \frac{\xi_n}{\Gamma(2-\beta_n)}x^{1-\beta_n} + \sum_{n=1}^\infty \frac{\eta_n}{\Gamma(2-\nu_n)}x^{1-\nu_n}, \qquad x > 0. \tag{22}$$

The function $\varkappa$ is monotone increasing. Denote

$$\varkappa^*(\zeta) = \limsup_{x\to\infty}\frac{\varkappa(\zeta x)}{\varkappa(x)}, \qquad \zeta > 1.$$

Since $\beta_n, \nu_n \in (0, 1)$, we find that

$$\varkappa(\zeta x) \leqslant \zeta\varkappa(x), \tag{23}$$

so that $\varkappa^*(\zeta) \leqslant \zeta$. Thus, $\varkappa$ is an O-regularly varying function (see [2], especially corollary 2.0.6).

On the other hand, it is seen from (22) that $\varkappa^*(\zeta) \geqslant 1$ for $\zeta > 1$, so that $\varkappa^*(+1) = 1$. Thus we are within the conditions of the 'ratio Tauberian theorem' ([2], theorem 2.10.1), which yields the relation $v_\lambda(x) \sim l\left(\frac{1}{x}\right)\varkappa(x)$, $x \to \infty$, so that

$$v_\lambda(x) \sim C\varkappa(x), \qquad x \to \infty. \tag{24}$$

In order to pass from (24) to (19), we have to check further Tauberian conditions. Since u_λ is completely monotone, it is, in particular, non-increasing, thus belonging to the class BI (see section 2.2 in [2] for the definitions of this class and the class PI used below). It follows from (23) that also $\varkappa \in \mathrm{BI}$.

Next,

$$\varkappa(x) \geqslant \sum_{n=0}^\infty \frac{\xi_n}{\Gamma(2-\beta_n)}x^{1-\beta_n} + \frac{\eta_1}{\Gamma(2-\nu_1)}x^{1-\nu_1}.$$

Since $\zeta^{1-\beta_n} \geqslant \zeta^{1-\nu_1}$ for $\zeta > 1$, we have

$$\begin{aligned}\frac{\varkappa(\zeta x)}{\varkappa(x)} &\geqslant \zeta^{1-\nu_1}\frac{\sum_{n=0}^\infty \frac{\xi_n}{\Gamma(2-\beta_n)}x^{1-\beta_n} + \frac{\eta_1}{\Gamma(2-\nu_1)}x^{1-\nu_1}}{\sum_{n=0}^\infty\left(\frac{\xi_n}{\Gamma(2-\beta_n)}x^{1-\beta_n} + \frac{\eta_n}{\Gamma(2-\nu_n)}x^{1-\nu_n}\right)}\\ &= \zeta^{1-\nu_1}\left\{1 - \frac{\sum_{n=2}^\infty \frac{\eta_n}{\Gamma(2-\nu_n)}x^{1-\nu_n}}{\sum_{n=0}^\infty\left(\frac{\xi_n}{\Gamma(2-\beta_n)}x^{1-\beta_n} + \frac{\eta_n}{\Gamma(2-\nu_n)}x^{1-\nu_n}\right)}\right\},\end{aligned}$$

where

$$\frac{\sum_{n=2}^{\infty} \frac{\eta_n}{\Gamma(2-\nu_n)} x^{1-\nu_n}}{\sum_{n=0}^{\infty} \left(\frac{\xi_n}{\Gamma(2-\beta_n)} x^{1-\beta_n} + \frac{\eta_n}{\Gamma(2-\nu_n)} x^{1-\nu_n} \right)} \leqslant C x^{-1+\nu_1} \sum_{n=2}^{\infty} \frac{\eta_n}{\Gamma(2-\nu_n)} x^{1-\nu_n} \longrightarrow 0,$$

as $x \to \infty$. Thus, $\varkappa \in \mathrm{PI}$.

Now we are within the conditions of the O-version of monotone density theorem ([2], proposition 2.10.3), which implies the required asymptotic relation (19).

(iii)–(iv) Let us prove (20). The proof of (21) is similar and simpler, and we leave it to the reader. It is obviously sufficient to deal with the first summand in each element of the series in (19).

Let us consider the function

$$\varphi(x) = x^{-a} (\log\log x)^b, \qquad x \geqslant e, \tag{25}$$

where $a, b > 0$. We have

$$\varphi'(x) = x^{-a} (\log\log x)^{b-1} \psi(x),$$

where $\psi(x) = \frac{b}{\log x} - a \log\log x$. It is easy to check that ψ decreases on $[e, \infty)$, $\psi(e) = b$, $\psi(x) \to -\infty$, as $x \to \infty$. The maximal value of the function φ is attained at a single point x_0 where $\psi(x_0) = 0$, that is

$$\frac{b}{\log x_0} = a \log\log x_0.$$

Denote $y_0 = \log x_0$. Then $\frac{b}{y_0} = a \log y_0$, so that $y_0 \log y_0 = \frac{b}{a}$. We will, in fact, need an asymptotic behavior of y_0 as a function of a, as $a \to 0$. It is known (see section I.5.2 in [8]) that

$$y_0 = \log \frac{b}{a} - \log\log \frac{b}{a} + O\left(\frac{\log\log \frac{b}{a}}{\log \frac{b}{a}} \right).$$

Therefore,

$$C_1 \frac{b}{a} \left(\log \frac{b}{a} \right)^{-1} \leqslant x_0 \leqslant C_2 \frac{b}{a} \left(\log \frac{b}{a} \right)^{-1}, \tag{26}$$

where the constants do not depend on a, b.

We have $\varphi(x) \leqslant \varphi(x_0)$, so that, by (25),

$$x^{-a} \leqslant \frac{\varphi(x_0)}{(\log\log x)^b}.$$

We need an estimate of $\varphi(x_0)$ making explicit its dependence on a. By (26),

$$x_0^{-a} \leqslant C_3 \left(\frac{b}{a} \right)^{-a} \left(\log \frac{b}{a} \right)^a,$$

where $\left(\frac{b}{a}\right)^{-a} = \mathrm{e}^{a \log \frac{b}{a}} \to 1$ and $\left(\log \frac{b}{a}\right)^a = \mathrm{e}^{a \log\log \frac{b}{a}} \to 1$, as $a \to 0$. Next, $x_0 \leqslant \frac{C_4}{a}$, so that

$$\log\log x_0 \leqslant \log \left(\log C_4 + \log \frac{1}{a} \right) \leqslant \log \left(C_5 \log \frac{1}{a} \right) \leqslant C_6 \log\log \frac{1}{a}$$

whence $\varphi(x_0) \leqslant C_7 \left(\log\log \frac{1}{a} \right)^b$.

As a result,

$$\sum_{n=0}^{\infty} \frac{\xi_n}{\Gamma(2-\beta_n)} x^{-\beta_n} \leqslant C_8 \left[\sum_{n=0}^{\infty} \xi_n \left(\log\log \frac{1}{\beta_n} \right)^b \right] \frac{1}{(\log\log x)^b},$$

and we have proved (20). □

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